THE EXTINCTION PROBABILITY IN A CRITICAL BRANCHING PROCESS

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HOWARD J. WEINER

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The Extinction Probability in a

Critical Branching Process

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I. Introduction.

Let $Z(t)$ denote the number of cells alive at t in a critical age-dependent Bellman-Harris branching process with cell lifetime distribution $G(t)$, $G(0+) = 0$, non-lattice and assume

$$t^2(1-G(t)) \to 0 \quad \text{as } t \to \infty . \tag{1.1}$$

$$0 < \mu \equiv \int_0^\infty t dG(t) < \infty . \tag{1.2}$$

Let the offspring generating function be denoted, for $0 \leq s \leq 1$,

$$h(s) \equiv \sum_{k=0}^{\infty} p_k s^k \tag{1.3}$$

and

$$h'(1) = 1 = \sum_{k=1}^{\infty} kp_k \qquad \text{(criticality)} \tag{1.4}$$

and

$$0 < \sigma^2 \equiv h''(1) = \sum_{k=2}^{\infty} k(k-1)p_k < \infty . \tag{1.5}$$

See [1] Chapter 4 for details.

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It is well-known that

$$\lim_{t \to \infty} tP[Z(t) > 0] = \frac{2\mu}{\sigma^2} \equiv b \; . \tag{1.6}$$

Various proofs of (1.6) and the corresponding result for a critical Galton-Watson or discrete time process have appeared. See [1] Chapters 1, 4 for comments and references. For example, the proof of (1.6) in [2] uses the renewal theorem. The proof in [1], Chapter 4, gives the asymptotic form for a critical generating function, from which (1.6) is a special case, and relates the generating function to that of a critical Galton-Watson process, for which results are obtained. The proof given in this note is elementary and self-contained.

II. Comparisons and Iterations

Definition. For $0 \leq s \leq 1$, $t \geq 0$,

$$F(s,t) \equiv \sum_{k=0}^{\infty} P[Z(t)=k]\, s^k. \tag{2.1}$$

It is well-known ([1], Chapter 4), and follows by the law of total probability that

$$F(s,t) = s(1-G(t)) + \int_0^t h(F(s,t-u))\,dG(u) \; . \tag{2.2}$$

Using the fact that

$$P(t) \equiv P[Z(t) > 0] = 1-F(0,t), \tag{2.3}$$

then from (2.2) it follows that, by a Taylor expansion on $h(s)$ about $s=1$, (2.4) $$1-P(t) = \int_0^t h(1-P(t-u))dG(u)$$

and

(2.5) $$1-P(t) = \int_0^t \{1-P(t-u) + \frac{\sigma^2}{2} P^2(t-u) + o(P^2(t-u))\} dG(u)$$

Rewriting (2.5),

(2.6) $$P(t) = 1-G(t) + \int_0^t P(t-u)dG(u) - \frac{\sigma^2}{2}\int_0^t P^2(t-u)dG(u) + f(t)$$

where $f(t)$ denotes the remainder.

Lemma 1 ([1]) $P(t)\downarrow 0$.

Proof. A simpler and elementary proof will be given here. Clearly

(2.7) $$P(t) \downarrow C \geq 0.$$

Assume $C>0$. Split the integral on the right side of (2.5) into $\int_0^t = \int_0^{t/2} + \int_{t/2}^t$. For large t, the integral $\int_{t/2}^t$ is $o(t^{-2})$. Then it is clear that

(2.8) $$1-C = 1-C+\frac{\sigma^2}{2} C + o(1),$$

which is a contradiction of $C>0$. This proves the lemma.

(2.9) Let $G^{(n)}(t) \equiv n^{th}$ iterate of G evaluated at t.

Define the iterative sequences

(2.10) $$U_{n+1}(t) = \int_0^t h(U_n(t-u)dG(u)$$

$$U_o(t) \equiv 1.$$

(2.11) $$I_{n+1}(t) = \int_0^t h(I_n(t-u))dG(u)$$

$$I_o(t) = \begin{cases} 1 \ , & t \le T \\ 1 - \frac{b}{t}, & t > T \end{cases}$$

for $T \ge b$.

Lemma 2 Under assumptions (1.1) - (1.5), for $n \ge 0$, and all $t > T$,

(2.12) $$0 \le U_n(t) - F(0,t) \le G^{(n)}(t)$$

(2.13) $$0 \le U_n(t) - I_n(t) \le G^{(n)}(t)$$

(2.14) $$|I_n(t)-I_o(t)| \le k(t),$$

where, for $t \to \infty$,

(2.15) $$tk(t) \to 0.$$

Proof. Eq. (212) holds for $n = 0$.

Assume (2.12) for n. Then, omitting arguments,

(2.16) $$0 \le U_{n+1}-F = \int_0^t h(U_n)-h(F)dG \le \int_0^t (U_n-F)dG$$

$$\le \int_0^t G^{(n)}(t-u)dG = G^{(n+1)}(t)$$

where the left inequality follows from the induction hypothesis and the monotonicity of h, and the right inequalities from the mean value theorem, the fact that $h'(1) = 1$, and the induction hypothesis.

To show (2.13), observe that for $t > T$

(2.17) $$0 \le U_o-I_o \le \frac{b}{t} < 1 = G^{(o)}(t) .$$

Assume (2.13) for n by induction. Then, arguing as in (2.16),

(2.18) $$0 \le U_{n+1}-I_{n+1} = \int_0^t (h(U_n)-h(I_n))dG \le \int_0^t (U_n-I_n)dG$$

$$\le \int_0^t G^{(n)}(t-u)dG(u) = G^{(n+1)}(t) .$$

To show (2.14) write $I_1(t)$ as

(2.19) $$I_1(t) = \int_0^{t/2} + \int_{t/2}^t h(I_o(t-u))dG(u) .$$

Note that the second integral ($\int_{t/2}^t$) in (2.19) is $o(t^{-2})$ by (1.1). The first integral ($\int_0^{t/2}$) may be written by a Taylor expansion of $h(s)$ about $s = 1$ as ($t \gg T$)

(2.20) $$I_1(t) = o(t^{-2}) + \int_0^{t/2} h(1-\frac{b}{t-u})dG(u)$$

$$= o(t^{-2}) + \int_0^{t/2} \{h(1) - (\frac{b}{t-u})h'(1) + \frac{1}{2}(\frac{b}{t-u})^2 h''(1) + o(t^{-2})\} dG(u)$$

Using $h(1) = h'(1) = 1$, and the expansion, $0 < u < t/2$,

(2.21) $$\frac{1}{t-u} = \frac{1}{t}(1 + \frac{u}{t} + o(\frac{u}{t}))$$

in the right side of (2.20) yields

(2.22) $$I_1(t) = o(t^{-2}) + G(t) - \frac{b}{t}\int_0^{t/2} (1 + \frac{u}{t} + o(\frac{u}{t}))dG(u)$$

$$+ \frac{b^2\sigma^2}{2t^2}\int_0^{t/2} \{1 + o(\frac{u}{t})\} dG(u)$$

as again by (1.1), an integration by parts yields

(2.23) $$\int_t^\infty u\,dG(u) = o(t^{-1}).$$

It follows that

(2.24) $$I_1(t) = I_0(t) + f(t)$$

where

(2.25) $$0 \leq |f(t)| \leq K < \infty$$

and as $t \to \infty$,

(2.26) $$t^2 f(t) \to 0.$$

Then one may write

(2.27) $$I_2(t) = \int_0^t h(I_0(t-u)+f(t-u))dG(u)$$

$$\geq \int_0^t \{h(I_0(t-u))+h'(I_0(t-u))f(t-u)\}dG(u).$$

Note that for $t \gg T$,

(2.28) $$I_1(T) = \int_0^T h(I_0(T-u))dG(u)$$

and

(2.29) $$h(1-\epsilon)G(T) \leq I_1(T) \leq h(1 - \frac{a}{T})G(T),$$

for some $a > 0$, $\epsilon > 0$,

(2.30) $$I_2(t) \leq \int_0^t \{h(I_0(t-u) + h'(\max\{h(1 - \frac{a}{T})G(T), 1 - \frac{\gamma}{t}\})f(t-u)\}dG(u)$$

for some $a > 0$, $\gamma > 0$.

For all $t \gg T$, (2.27) - (2.30) yield that

$$|I_2(t)-I_1(t)| \leq (1 - \frac{\alpha}{t})\int_0^t f(t-u)dG(u). \tag{2.31}$$

Hence, repeating the argument of (2.27) - (2.31),

$$|I_3(t)-I_2(t)| \leq (1 - \frac{\alpha}{t})^2 \int_0^t f(t-u)dG^{(2)}(u), \tag{2.32}$$

where

$$0 < \alpha \text{ is a constant.} \tag{2.33}$$

An induction yields that

$$|I_n(t)-I_0(t)| \leq \sum_{\ell=0}^{\infty} (1 - \frac{\alpha}{t})^{\ell} f{*}G^{(\ell)}(t) \equiv k(t), \tag{2.34}$$

where "$*$" denotes the usual convolution integral.

The right side of (2.34) is now broken up into a number of parts, and upper bounds for each part is obtained.

$$k(t) = \sum_{\ell=0}^{t^2} + \sum_{\ell=t^2}^{\infty} (1 - \frac{\alpha}{t})^{\ell} f{*}G^{(\ell)}(t). \tag{2.35}$$

Since

$$|f|{*}G^{(\ell)}(t) \leq K \tag{2.36}$$

the second term of (2.35) is dominated by

$$K \sum_{\ell=t^2}^{\infty} (1 - \frac{\alpha}{t})^{\ell} \leq \frac{Kte^{-\alpha t}}{\alpha}. \tag{2.37}$$

The first term on the right of (2.35) is now written as

$$\sum_{\ell=0}^{t^2} (1 - \tfrac{\alpha}{t})^{\ell} f{*}G^{(\ell)}(t) \tag{2.38}$$

$$= \sum_{\ell=0}^{t^2} (1 - \tfrac{\alpha}{t})^{\ell} \left[\int_0^{t-t/\ln t} + \int_{t-t/\ln t}^{t} f(t-u)dG^{(\ell)}(u)\right].$$

The first term on the right side of (2.38) is bounded above by

$$\frac{t}{\alpha}\, o\!\left(\frac{(\ln t)^2}{t^2}\right) = o(t^{-1}), \tag{2.39}$$

by the asymptotic behavior of f and summing the geometric series.

The second term on the right side of (2.38) is split into the three terms, ignoring distinctions between $[c]$ and c,

$$\sum_{\ell=0}^{\sqrt{t/\ln t}} + \sum_{\ell=\sqrt{t}/\ln t}^{t \ln t} + \sum_{\ell=t \ln t}^{t^2} \int_{t-t/\ln t}^{t} f(t-u)dG^{(\ell)}(u). \tag{2.40}$$

Since $|f| \le 1$, Chebyshev's inequality yields

$$\int_{t-t/\ln t}^{t} |f(t-u)|dG^{(\ell)}(u) \le \frac{a\ell}{(t-t/\ln t - \ell\mu)^2}, \tag{2.41}$$

where

$$0 < a^2 = \int_0^{\infty} (t-\mu)^2 dG(u). \tag{2.42}$$

An application of (2.41) to the first term of (2.40) yields

$$\left|\sum_{\ell=0}^{\sqrt{t/\ln t}} \int_{t-t/\ln t}^{t} f(t-u)dG^{(\ell)}(u)\right| \le \frac{\beta}{t(\ln t)^2} \tag{2.43}$$

for some $\beta > 0$.

The Central Limit theorem may be applied to the second and third terms of (2.40) since the summation index ℓ is large.

For $\ell \geq \sqrt{t}/\ln t$,

$$G^{(\ell)}(t) - G^{(\ell)}(t-t/\ln t) \sim \Phi\left(\frac{t-\ell\mu}{a\sqrt{\ell}}\right) - \Phi\left(\frac{t-t/\ln t - \ell\mu}{a\sqrt{\ell}}\right). \tag{2.44}$$

Applying (2.44) to the second term of (2.40) and using the mean value theorem and

$$\varphi(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2}, \tag{2.45}$$

yields

$$\left| \sum_{\ell=\sqrt{t}/\ln t}^{t \ln t} \int_{t-t/\ln t}^{t} f(t-u)\,dG^{(\ell)}(u) \right| \tag{2.46}$$

$$\leq \sum_{\ell=\sqrt{t}/\ln t}^{t \ln t} \varphi\left(\frac{t-\ell\mu}{a\sqrt{\ell}}\right)\left(\frac{t}{a(\ln t)\sqrt{\ell}}\right)$$

$$\leq Ce^{-t/2\ln t}\left(\frac{2t^{3/2}}{a \ln t}\right),$$

where C is a positive constant.

The Central Limit theorem is applied to the third term of (2.40), and since the arguments are large, use will be made of the standard approximation

$$1 - \Phi(x) \sim \frac{\beta}{x}\,\varphi(x) \tag{2.47}$$

for some $\beta > 0$, as $x \to \infty$.

Applying (2.44) and (2.47) to the third term of (2.40) yields

$$\left| \sum_{\ell=t\,\ln t}^{t^2} \int_{t-t/\ln t}^{t} f(t-u)\,dG^{(\ell)}(u) \right| \tag{2.48}$$

$$\leq \beta \sum_{\ell=t\,\ln t}^{t^2} \varphi\left(\frac{t-\ell\mu}{a\sqrt{\ell}}\right)\left[\frac{a\sqrt{\ell}}{t-t/\ln t\ -\ \ell\mu} - \frac{a\sqrt{\ell}}{t-\ell\mu}\right]$$

$$\leq \frac{\beta t^2 \varphi\left(\frac{-\mu\sqrt{t\ \ln t}}{a}\right)(at)(t/\ln t)}{t^4 \mu^2} \leq \frac{\gamma e^{-\delta t\ \ln t}}{\ln t}\ .$$

Now (2.37), (2.39), (2.43), (2.46), (2.48) applied to (2.34) yield that for all sufficiently large t, equation (2.14) holds.

Theorem 1. *Under assumptions* (1.1) *to* (1.5)

$$\lim_{t\to\infty} tP[Z(t) > 0] = b. \tag{1.6}$$

Proof. Combine (2.12) - (2.15) to yield

$$\left|P[Z(t) > 0] - \frac{b}{t}\right| \leq G^{(n)}(t) + o(t^{-1}). \tag{2.49}$$

Let $n \to \infty$. The weak law of large numbers yields the result.

III. Extension. The result of Theorem 1 can be strengthened by the method.

Theorem 2. *Under the assumptions* (1.2) - (1.5) *and in addition, for* $t \to \infty$,

$$t^3(1-G(t)) \to 0 \tag{3.1}$$

and

$$(3.2) \qquad h^{(3)}(1) < \infty,$$

then for t *large,*

$$(3.3) \qquad P[Z(t) > 0] \sim \frac{b}{t} + \frac{c \ln t}{t^2}$$

for some unspecified constant c.

Remark. Conditions (3.1) and (3.2) are not as stringent as in [2], where more terms in the asymptotic expansion of $P(t)$ are given.

Outline of Proof.

Define

$$(3.4) \qquad J_{n+1}(t) = \int_0^t h(J_n(t-u)dG(u)$$

$$J_0(t) = \begin{cases} 1, & t \leq T \\ 1 - \frac{b}{t} - \frac{c \ln t}{t^2}, & t > T \end{cases}$$

for $T \gg b$.

As in the proof of (2.13) one obtains

$$(3.5) \qquad 0 \leq U_n(t) - J_n(t) \leq G^{(n)}(t).$$

An expansion of h about 1 to four terms in the Taylor expansion yields that

$$(3.6) \qquad J_1(t) = J_0(t) + o(t^{-3}).$$

From this and a similar tedious sequence of estimations as in Theorem 1, one obtains

$$|J_n(t)-J_0(t)| \leq G^{(n)}(t) + o(t^{-2}) \tag{3.7}$$

and again one lets $n \to \infty$ and applies the weak law of large numbers to complete the argument.

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[1] ATHREYA, K. B. and NEY, P. E. (1970) Branching Processes Springer-Verlag, New York

[2] CHOVER, J. and NEY, P. E. (1968) The non-linear renewal equation. J. d'Analyse Math. 21, 381-413.

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Let $Z(t)$ be the number of cells alive at t in a critical age-dependent Bellman-Harris process with lifetime distribution $G(t)$ and offspring generating function $h(s)$. A simplified proof that $tP(Z(t) > 0) \to b > 0$, a specified constant, is given.

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